One-Dimensional Cellular Automata, Conservation Laws and Partial Differential Equations

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The existence of conservation laws (invariants) are discussed for various one-dimensional cellular automata. The cellular automata are derived from partial differential equations. Both irreversible and reversible cellular automata are investigated.

Key words: Cellular Automata; Conservation Laws; Partial Differential Equations; Spin Systems.

Cellular automata (see [1 – 4]) may be considered as discrete dynamical systems. They are discrete in several aspects. First, they consist of a discrete spatial lattice of sites. Second they evolve in discrete time steps, i. e., $t = 0, 1, 2 \ldots$. Third each lattice site j (or box, or cell) has only a finite discrete set of possible values. The simplest case is that the dependent variable u_j takes two values, say 0 and 1, i. e., in modulo 2 integer arithmetic we have $0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$, $1 + 1 = 0$. The simplest case of a lattice is a linear chain with N lattice sites (or boxes).

Any cellular automata rule can be described by an evolution equation of the form

$$u_j(t+1) = F[u_{\{j\}}(t)], \tag{1}$$

where $u_j(t)$ is the state of the site j at time t and $F[u_{\{j\}}(t)]$ is a function of the states of sites in a neighbourhood of lattice site j at time t. The one-dimensional case (i. e., one has a linear chain) (1) can also be written as

$$u_j(t{+}1) = F[u_{j-r}(t), u_{j-r+1}(t), \ldots, u_{j+r}(t)]. \tag{2}$$

The local rule F has the range of r sites. Numerical studies suggest that the pattern generated in the time evolution of cellular automata from disordered initial states can be classified as follows:

(i) evolves to homogeneous state;

(ii) evolves to simple separated periodic structures;

(iii) evolves to chaotic aperiodic patterns;

(iv) evolves to complex pattern of localized structures.

Constants of motion (if any exist) in classical mechanics and conservation laws (if any exist) in field theory play an important role in studying the behaviour of the time evolution. The same holds for cellular automata. If conservation laws (also called invariants) exist for a given cellular automaton, then one has a partition of its state space. We study the existence of conservation laws for various cellular automation. Both irreversible and reversible cellular automata are discussed here. A rule of a cellular automaton is said to be reversible if it is backwards deterministic. In the following we assume that $j = 0, 1, \ldots, N-1$, and we apply periodic (cyclic) boundary conditions, i. e., $N \equiv 0$. Obviously if N is finite the pattern generated can only be (i) or (ii).

The most studied cellular automaton is given by

$$u_j(t+1) = [u_{j-1}(t) - u_{j+1}(t)] \bmod 2, \tag{3}$$

where $t = 0, 1, 2, \ldots$ and u_j takes two values, 0 and 1. Equation (3) corresponds to rule 90. The map given by (3) is irreversible. Since each cell can take two values the number of initial configurations is given by 2^N. Let $N = 2^5 = 32$ with the initial configuration given by $u_j(0) = 0$ for $j = 0, \ldots, 14, 16, \ldots, 31$ and $u_{15}(0) = 1$. The number of time steps is 20. We find that $u_j(t) = 0$ for all $j = 1, \ldots, 31$ at $t \geq 16$. Thus the cellular automaton tends to the fixed point $u_0^* = u_1^* = \cdots = u_{31}^* = 0$. In other words the cellular automaton evolves to a homogeneous state. Let $N = 31$ with the initial configuration given by $u_j(0) = 0$ for $j = 0, 1, \ldots, 14, 16, \ldots, 30$ and $u_{15}(0) = 1$. The number of time steps is 60. We find the case (ii), i. e., the pattern has a simple separated periodic structure. Thus the behaviour depends on the number of lattice sites, i. e., whether N is even or odd.

To discuss the conservation laws for the cellular automaton (3) we make a change of the variables from u_j to the (spin)-variables $s_j = -1, +1$, where $s_j = -1$ corresponds to $u_j = 0$ and $s_j = +1$ corresponds to $u_j = 1$. Then the map (3) takes the form

$$s_j(t+1) = -s_{j-1}(t)s_{j+1}(t). \tag{4}$$

The state with all $s_j = -1$ is an invariant state (fixed point) of the mapping (4). Now we find that

$$s_0(t)s_1(t)\cdots s_{N-1}(t) = (-1)^N \tag{5}$$

for $t = 1, 2, \dots$, since $(s_j(t))^2 = 1$ and $s_0(t) \equiv s_N(t)$. This implies that no state (N-spin configuration) with an odd number of up spins $(+1)$ is present at any time in the evolution except possibly at the initial time $t = 0$. Furthermore it can be proved that

$$s_j(t+2^n) = -s_{j-2^n}(t)s_{j+2^n}(t) \tag{6}$$

for $t = 0, 1, 2, \dots$ and $n = 0, 1, 2, \dots$. For $N = 2^k, k = 1, 2, \dots$ one has all $s_j(t) = -1$ [i. e., $u_i(t) = 0$] for all times $t \geq 2^{k-1}$ irrespective of the initial spin configuration of the ring.

A large class of cellular automata is provided by discretization (of space and time coordinates) of partial differential equations. A simple example is the one-dimensional linear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{7}$$

with the conserved quantity

$$\int_{\mathbf{R}} u(x,t)\mathrm{d}x = C, \tag{8}$$

where C is the total amount of the diffusing substance. Here it is assumed that u and its derivative with respect to x go to zero as $|x| \to \infty$. The simplest discretization of the one-dimensional diffusion equation is given by

$$u_j(t+1) - u_j(t) = u_{j+1}(t) - 2u_j(t) + u_{j-1}(t) \tag{9}$$

with unit discretization steps. In modulo 2 integer arithmetic the term $2u_j$ vanishes. Therefore

$$u_j(t+1) = u_{j+1}(t) + u_j(t) + u_{j-1}(t) \bmod 2. \tag{10}$$

Equation (10) corresponds to rule 150. Again we impose periodic boundary conditions. As a conservation law we find

$$\sum_{j=0}^{N-1} u_j(t) = C, \tag{11}$$

where the constant C is given by

$$C = \sum_{j=0}^{N-1} u_j(0). \tag{12}$$

Note that C can only take the values 0 or 1. The proof is straightforward. We take the sum over j of the left- and right-hand side of (10) and bear in mind that we have modulo 2 integer arithmetic and cyclic boundary conditions. Let $N = 2^5 = 32$ with the initial configuration $u_j(0) = 0$ for $j = 0, 1, \dots, 14, 16, \dots, 31$ and $u_{15}(0) = 1$. The number of time steps is 40. The system evolves to a periodic structure. Let $N = 31$ with the initial configuration $u_j(0) = 0$ for $j = 0, 1, \dots, 14, 16, \dots 30$ and $u_{15}(0) = 1$. Again the system evolves to a periodic structure.

As a third example we consider the Burgers equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}. \tag{13}$$

The equation can be written as a conservation law:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(-\frac{u^2}{2} + \frac{\partial u}{\partial x}\right) \tag{14}$$

with the conserved quantity

$$\int_{\mathbf{R}} u(x,t)\mathrm{d}x = C. \tag{15}$$

Here we have assumed that u and its derivate with respect to x go to zero as $|x| \to \infty$. The simplest discretization yields

$$\begin{aligned} u_j(t+1) - u_j(t) = {} & u_{j+2}(t) - 2u_{j+1}(t) + u_j(t) \\ & - u_j(t)[u_{j+1}(t) - u_j(t)] \end{aligned} \tag{16}$$

with unit discretization steps. In modulo 2 integer arithmetic we have $u_j u_{j+1} = -u_j u_{j+1}$ and $u_j = u_j^2$ for all t. Therefore (16) simplifies to

$$\begin{aligned} u_j(t+1) = {} & u_{j+2}(t) + u_j(t)u_{j+1}(t) \\ & + u_j(t) \bmod 2. \end{aligned} \tag{17}$$

We see that $u_j(t) = 0$ for all $i = 0, 1, \ldots, N-1$ and $u_j(t) = 1$ for all $j = 0, 1, \ldots, N-1$ are invariant states. We see that $\sum_{j=0}^{N-1} u_j(t) = C$ is no longer a conservation law. Let $N = 32$ and the initial configuration $u_j(0) = 0$ for $j = 0, 1, \ldots, 14, 16, \ldots, 31$ and $u_{15}(0) = 1$. The number of time steps is 20. The cellular automaton tends to the fixed point $u_0^* = u_1^* = \cdots = u_{31}^* = 0$. Let $N = 31$ and the initial configuration $u_j(0) = 0$ for $j = 0, 1, \ldots, 14, 16, \ldots, 30$ and $u_{15}(0) = 1$. The pattern reached has a simple periodic structure.

The Korteweg – de Vries equation is given by

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\,. \tag{18}$$

The Korteweg – de Vries equation has an infinite number of conservation laws [5]. The conserved density for the first three are

$$\begin{aligned} T_1(u) &= u, \\ T_2(u) &= u^2, \\ T_3(u) &= u^3 - 3(\partial u/\partial x)^2. \end{aligned} \tag{19}$$

Since the Korteweg – de Vries equation can be written as a conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{u^2}{2} - \frac{\partial^2 u}{\partial x^2}\right) = 0, \tag{20}$$

it follows that $T_1(u) = u$ is a conserved density with the conserved current given by $X(u) = -u^2/2 - \partial^2 u/\partial x^2$. After the simplest discretization we have

$$\begin{aligned} u_j(t+1) - u_j(t) &= u_j(t)(u_{j+1}(t) - u_j(t)) \\ &+ u_{j+2}(t) - 3u_{j+1}(t) + 3u_j(t) - u_{j-1}(t). \end{aligned} \tag{21}$$

Since in modulo 2 arithmetic we have $3u_j(t) = u_j(t)$ and $2u_j(t) = 0$ we obtain the cellular automata

$$\begin{aligned} u_j(t+1) = u_j(t)(u_{j+1}(t) - u_j(t)) + u_{j+2}(t) \\ - u_{j+1}(t) - u_{j-1}(t). \end{aligned} \tag{22}$$

Thus $\sum_{j=0}^{N-1} u_j(t)$ is not an invariant.

Let us now consider reversible cellular automata. A rule of a cellular automaton is said to be reversible if it is backwards deterministic. There is a simple way to construct reversible rules. Take any rule F involving n states per cell, note the value it returns and subtract from it, in modulo n arithmetic, the value that the centre cell assumed at time $t-1$:

$$u_j(t+1) = F[u_{\{j\}}] - u_j(t-1) \bmod n. \tag{23}$$

This relation can be solved uniquely for $u_i(t-1)$, even if F is not invertible. Thus the rule is reversible. The rules constructed above are not only reversible, they are also time reversal invariant. A sequence of configurations can be obtained in reverse order with the same rule, simply by inverting the last two configurations. However, not all rules are invariant under time reversal. In general, a reversible cellular automaton has as many conserved quantities as there are cells. It remembers the initial state of each cell since one can recover this information by running the system backwards.

As an example let us consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = u. \tag{24}$$

The Hamiltonian density is given by

$$\mathcal{H}(u, u_t, u_x) = \frac{1}{2}[(u_t)^2 + (u_x)^2] - \frac{1}{2}u^2, \tag{25}$$

which is a conserved quantity. The simplest discretization of (24) yields

$$\begin{aligned} u_j(t+1) - 2u_j(t) + u_j(t-1) - u_{j+1}(t) \\ + 2u_j(t) - u_{j-1}(t) = u_j(t). \end{aligned} \tag{26}$$

In modulo 2 arithmetic we obtain the reversible cellular automaton

$$\begin{aligned} u_j(t+1) = u_{j+1}(t) + u_j(t) \\ + u_{j-1}(t) + u_j(t-1), \end{aligned} \tag{27}$$

where $t = 0, 1, \ldots$. The map is of second order. If $N = 32$ and the initial configuration is $u_j(-1) = u_j(0) = 0$ for $j = 0, 1, \ldots, 14, 16, \ldots, 30$, $u_{15}(-1) = u_{15}(0) = 1$ we find a periodic pattern. Also for $N = 31$ and the initial configuration $u_j(-1) = u_j(0) = 0$ for $j = 0, 1, \ldots, 14, 16, \ldots, 30$ and $u_{15}(-1) = u_{15}(0) = 1$ we find a periodic pattern.

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